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NOTES ON OPEN CHANNEL FLOW - UNSTEADY FLOW

Profili di moto permanente in un canale e in una serie di due canali - Boudine, 1861
OPEN CHANNEL FLOW: when in unsteady state?

Studying flood propagation and hydraulic hazard mapping

Unsteady operative conditions of hydraulic works (e.g., surges)
OPEN CHANNEL FLOW: 1D and 2D flow
OPEN CHANNEL FLOW: basic hypothesis of unsteady flow

Same hypothesis introduced for steady flow:

1D flow
Gradually varied flow (no abrupt geometrical transition; negligible vertical acceleration)
Small bed slope
Fixed bed; density $\rho$ is constant and the flow regime is turbulent, mostly hydraulically rough
$\alpha$ and $\beta = 1$
$S_f$ evaluated as in steady flow
OPEN CHANNEL FLOW: Mass balance for 1D flow

\[ M = \int_{w} \rho dW \]

\[ \frac{DM}{Dt} = \frac{D}{Dt} (\int_{w} \rho dW) = 0 \]

\[ \frac{D}{Dt} (\int_{w} \rho dW) = \frac{\partial}{\partial t} (\int_{w} \rho dW) - \int_{s} \rho (\nabla \cdot \mathbf{n}) dS = \frac{\partial}{\partial t} (\int_{w} \rho dW) - \int_{s} \rho dQ = 0 \quad (1) \]

Mass balance for a 1D flow on an infinitesimal stretch \( dx \), rewriting eq. (1)

\[ \frac{\partial}{\partial t} (\int_{w} \rho dW) = \rho \frac{\partial}{\partial t} \int_{s} dSdx = \rho \frac{\partial S}{\partial t} dx \]

\[ \int_{s} \rho dQ = \rho Q(x) - \rho Q(x + dx) = -\rho \frac{\partial Q}{\partial x} dx \]

\[ \frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (1a) \]

or, equivalently, given that \( S = S(h(t)) \)

\[ b(h) \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (1b) \]
**OPEN CHANNEL FLOW: Momentum balance for 1D flow**

\[
\frac{D}{Dt} \int_{W} \rho \bar{v} dW = \int_{W} \rho \bar{g} dW + \int_{S} \bar{\sigma}_n dS
\]

\[-\frac{\partial}{\partial t} (\int_{W} \rho \bar{v} dW) + \int_{S} \rho \bar{v} (\bar{v} \cdot \bar{n}) dS + \int_{W} \rho \bar{g} dW + \int_{S} \bar{\sigma}_n dS = 0 \quad (2)
\]

This is a vectorial balance

Scalar component of the momentum balance for a 1D flow in the prevailing direction of flow, on an infinitesimal stretch \(dx\), rewriting eq. (2)

\[I = \rho \frac{\partial Q}{\partial t} \quad M = \rho \beta Q U = \rho Q U\]

\[\Pi = \int_{0}^{h} \gamma (h - \eta) b(\eta) d\eta = \gamma A\]

\[\Pi_{3x} = \int_{S} \gamma (h - \eta) \frac{\partial b}{\partial x} d\eta = \gamma B\]

\[-Idx - \frac{dM}{dx} dx - \frac{d\Pi}{dx} dx + \Pi_{3x} dx - \tau_0 P dx + \gamma \text{sen}(\alpha) dx = 0\]

\[-\frac{\partial Q}{\partial t} - \frac{\partial}{\partial x} \left( \frac{\beta Q^2}{S} \right) - g \frac{\partial I_A}{\partial x} + gI_B - \frac{\tau_0 P}{\rho} + g \text{sen}(\alpha) = 0\]

\[-\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{S} + gI_A \right) = +gI_B - \frac{\tau_0 P}{\rho} + g \text{sen}(\alpha) = 0\]

\[-\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{S} + gI_A \right) = g(Sb - Sf) + gI_B = 0\]

\[(2a)\]
Momentum balance for a 1D flow on an infinitesimal stretch $dx$, rewriting eq. (1) + (2)

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{S} + gI_A \right) = gS(S_b - S_f) + gI_B = 0
\]

\[
\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = 0
\]

This system of equations can be written also in vectorial form as

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} = \mathbf{S}
\]

\[
\mathbf{U} = \begin{bmatrix} S \\ Q \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} Q \\ \frac{Q^2}{S} + gI_A \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ gS(S_b - S_f) + gI_B \end{bmatrix}
\]

where $S$ is the so-called source term.

These are the so-called De Saint Venant equations (1871) or Shallow Water Equations.
OPEN CHANNEL FLOW: Observations on the 1D de Saint Venant Equations

\[ \frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = 0 \]  
\[ (1a) \]

\[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{S} + gI_A \right) = gS( S_b - S_f ) + gI_B = 0 \]  
\[ (2a) \]

Using Leibniz integral rule, one can observe that

\[ g \frac{\partial}{\partial x} I_A = \frac{\partial}{\partial x} \int_0^h g(h - \eta)b(\eta)d\eta = \frac{\partial}{\partial x} \int_0^h \frac{\partial h}{\partial \eta} b(\eta)d\eta = \frac{\partial h}{\partial \eta} b(\eta)d\eta = gS \frac{\partial h}{\partial x} \]

Accordingly, eq. (2a) can be written as

\[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{S} \right) + gS \frac{\partial h}{\partial x} = gS( S_b - S_f ) \]

Let us now consider a steady state situation, with constant discharge along the river:

\[ \frac{\partial Q}{\partial t} = 0; \quad \frac{\partial Q}{\partial x} \equiv \frac{dQ}{dx} = 0 \]

Accordingly, the momentum balance can be written as:

\[ - \frac{Q^2}{S^2} \frac{dS}{dx} + gS \frac{dh}{dx} = gS( S_b - S_f ) \]

If the channel is prismatic, \( S=S(h(x)) \), so that:

\[ \frac{dh}{dx} = \frac{S_b - S_f}{1 - \frac{Q^2}{gS^3} b} = \frac{S_b - S_f}{1 - Fr^2} \]

that has been used to study steady water surface profiles.
If the channel has a rectangular cross-section with constant width $B$, $S=Bh$; $Q=Uh$. The continuity equation provides

$$\frac{\partial h}{\partial t} + \frac{\partial Uh}{\partial x} = 0 \quad (1c)$$

and the momentum equation provides

$$b \frac{\partial Uh}{\partial t} + \frac{\partial}{\partial x} \left( U^2 bh + \frac{1}{2} gbh^2 \right) = gbh \left( S_b - S_f \right)$$

$$U \frac{\partial h}{\partial t} + h \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left( U^2 h + \frac{1}{2} gh^2 \right) = gh \left( S_b - S_f \right)$$

$$-U \frac{\partial Uh}{\partial x} + h \frac{\partial U}{\partial t} + \frac{\partial U^2 h}{\partial x} + gh \frac{\partial h}{\partial x} = gh \left( S_b - S_f \right)$$

$$-U \frac{\partial Uh}{\partial x} + h \frac{\partial U}{\partial t} + U \frac{\partial Uh}{\partial x} + Uh \frac{\partial U}{\partial x} + gh \frac{\partial h}{\partial x} = gh \left( S_b - S_f \right)$$

$$\frac{1}{g} \frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left( \frac{U^2}{2g} + \frac{\partial h}{\partial x} \right) = \left( S_b - S_f \right) \quad (2c)$$

That in steady state can be written as

$$\frac{\partial H}{\partial x} = -S_f; \quad \frac{\partial E}{\partial x} = S_b - S_f$$
We have seen that an infinitesimal wave that involves the whole flow propagates with a celerity relative to the flow

\[ c = \sqrt{gh_m} = \sqrt{g \frac{S(h)}{b(h)}} \]

and with an absolute celerity relative to the river's banks

\[ U \pm c = U \pm \sqrt{g \frac{S(h)}{b(h)}} = U \pm \sqrt{gh_m} = \sqrt{gh_m} \left( \frac{U}{\sqrt{gh_m}} \pm 1 \right) = \sqrt{gh_m} (Fr \pm 1) \]

Accordingly, the infinitesimal wave propagates in space along the path \( x(t) \) given by equation

\[ \frac{dx}{dt} = U \pm \sqrt{g \frac{S(h)}{b(h)}} = \sqrt{gh_m} (Fr \pm 1) \]

where \( x \) is positive in the direction of the river main flow. Accordingly, we have two different directions that depend on the value of the Froude number. More specifically:

- \( Fr < 1 \)
- \( Fr = 1 \)
- \( Fr > 1; U > 0 \)
- \( |Fr| > 1; U < 0 \)

Where \( h_m \) is the average depth in the cross-section.
Where the area between the two lines is the so-called “analytic dependence domain”, that is the region in space that is directly influenced by what happens in the river stretch AB.

The physical condition in P depends only on the flow status in AB.

In a similar way, one can identify the region in space that is influenced by what happens in Q (range of influence).
Accordingly, information does not propagate isotropically in space-time; for instance, 

In order to compute \((U,h)\) in \(P\), \((U_A,h_A)\) and \((U_B,h_B)\) are needed. These informations are located upstream and downstream of \(P\) if \(Fr < 1\) and only upstream in \(Fr > 1\).

As one can understand, this strongly conditions the whole physics of the process and, accordingly, also every attempt of its solution (both by analytic and numeric point of view).

For instance, it would not make sense to compute numerically \((U_P,h_P)\) using the a numerical approach based on the information in \(B\) and \(D\). On the contrary, an “upwind” method (that follows the characteristic directions) is needed.
In the same way, boundary conditions (BC), which can be seen as a superimposition of infinitesimal perturbation to the uniform flow, will propagate within the domain following the characteristic lines. They must be provided (blue lines beside), in addition to the initial state of the system at \( t=0 \) (red line beside), in order to solve the problem on the whole \((x,t)\) domain.

Accordingly, if \( Fr < 1 \) a BC must be given upstream (e.g., the value of \( Q \) flowing in a river) and another must be given downstream (e.g., a water level). If \( Fr > 1 \) both BCs must be given upstream.

In general, the number and location of BC must be equal to the number of characteristic lines entering the domain.

Tipically, if \( Fr < 1 \), \( BC(0,t)=Q(t) \); \( BC(L,t)=h(t) \);

In place of \( h(t) \), one can give a stage-discharge relationship \( Q(h(t)) \); These BC are both given upstream if \( Fr > 1 \)
OPEN CHANNEL FLOW: theory of characteristic lines

These considerations pave the way to a different statement of the De Saint Venant equations, that eventually transform them from PDE to ODE.

For simplicity’s sake, let us consider a rectangular channel with width $b$

$$\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial U h}{\partial x} &= 0 \quad (1c) \\
 b \frac{\partial U h}{\partial t} + \frac{\partial}{\partial x} \left( U^2 b h + gb \frac{h^2}{2} \right) &= gbh \left( S_b - S_f \right) \quad (2c)
\end{align*}$$

These equations can be rewritten as

$$\begin{align*}
\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} &= 0 \quad (1d) \\
\text{and} \quad U \frac{\partial h}{\partial t} + h \frac{\partial U}{\partial x} + U^2 \frac{\partial h}{\partial x} + h \frac{\partial U^2}{\partial x} + gh \frac{\partial h}{\partial x} &= gh \left( S_b - S_f \right) \\
U \left( \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} \right) + h \frac{\partial U}{\partial t} + hU \frac{\partial U}{\partial x} + gh \frac{\partial h}{\partial x} &= gh \left( S_b - S_f \right) \\
\frac{\partial U}{\partial t} + hU \frac{\partial U}{\partial x} + gh \frac{\partial h}{\partial x} &= gh \left( S_b - S_f \right)
\end{align*}$$
OPEN CHANNEL FLOW: theory of characteristic lines

\[ + \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial h}{\partial x} = g \left( S_b - S_f \right) \equiv \bar{E} \quad (2d) \]

Let us operate a change of dependent variables

\[ c = \sqrt{gh} \]
\[ h = \frac{c^2}{g} \]

From which one gets

\[ \frac{\partial h}{\partial x} = 2 \frac{c}{g} \frac{\partial c}{\partial x} \]
\[ \frac{\partial h}{\partial t} = 2 \frac{c}{g} \frac{\partial c}{\partial t} \]

Which can be substituted within (1d) and (2d) obtaining

\[ 2 \frac{\partial c}{\partial t} + 2U \frac{\partial c}{\partial x} + c \frac{\partial U}{\partial x} = 0 \quad (1e) \]
\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + 2c \frac{\partial c}{\partial x} = \bar{E} \quad (2e) \]
And considering (1e) + (2e)

\[
\frac{\partial 2c}{\partial t} + (U + c) \frac{\partial 2c}{\partial x} + \frac{\partial U}{\partial t} + (U + c) \frac{\partial U}{\partial x} = E
\]  

(3a)

and (1e) - (2e)

\[
\frac{\partial 2c}{\partial t} + (U - c) \frac{\partial 2c}{\partial x} - \left[ \frac{\partial U}{\partial t} + (U - c) \frac{\partial U}{\partial x} \right] = -E
\]  

(3b)

Which can be written as

\[
\begin{align*}
\frac{D(U + 2c)}{Dt} &= E \\
\frac{dx}{dt} &= U + c & C^+ \\
\frac{D(U - 2c)}{Dt} &= E \\
\frac{dx}{dt} &= U - c & C^- 
\end{align*}
\]
Let us consider the De Saint Venant equations written for a rectangular prismatic channel with width $b$

$$\begin{align*}
\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + \frac{1}{g} \frac{dU}{dt} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial x} \left( \frac{U}{g} \frac{dU}{dh} + 1 \right) &= 0 \\
\frac{\partial}{\partial x} \left( \frac{U}{g} \frac{dU}{dh} \right) + \frac{\partial h}{\partial x} \left( \frac{U}{g} \frac{dU}{dh} + 1 \right) &= 0
\end{align*}$$

By eliminating the derivative of $h$ with respect to time, one gets

$$\frac{\partial h}{\partial x} \left[ 1 - \frac{h}{g} \left( \frac{dU}{dh} \right)^2 \right] = 0 \quad \text{that is} \quad \frac{dU}{dh} = \pm \sqrt{\frac{g}{h}}$$
has the general integral
\[ \frac{dU}{dh} = \pm \sqrt{\frac{g}{h}} \]

From that equation one can observe that
\[ a = U + \sqrt{gh} \quad \rightarrow \quad dU = +\sqrt{\frac{g}{h}} dh \]
\[ a = U - \sqrt{gh} \quad \rightarrow \quad dU = -\sqrt{\frac{g}{h}} dh \]

\[ U \mp 2\sqrt{gh} = \text{Cost}; \]

\[ \frac{\partial h}{\partial t} + \left( U + \frac{dU}{dh} \right) \frac{\partial h}{\partial x} = 0 \]

\[ \frac{\partial h}{\partial t} + (U \pm \sqrt{gh}) \frac{\partial h}{\partial x} = 0 \]

\[ \frac{Dh}{Dt} = 0 \]  \[ a = U \pm \sqrt{gh} \]
In conclusion:

\[ dU + \sqrt{\frac{g}{h}}dh \rightarrow U - 2\sqrt{gh} = \text{Cost} \rightarrow a = U + \sqrt{gh} \]

\[ dU - \sqrt{\frac{g}{h}}dh \rightarrow U + 2\sqrt{gh} = \text{Cost} \rightarrow a = U - \sqrt{gh} \]

Let us now consider a still water body where \( U=0 \) and \( Y = Y_0 \). Let us consider the wave that propagates with positive relative velocity: the general integral will be

\[ U - 2\sqrt{gh} = 0 - 2\sqrt{gh_0} \]

And the corresponding absolute celerity will be

\[ a = U + \sqrt{gh} \equiv 3\sqrt{gh} - 2\sqrt{gh_0} \]

Accordingly, if \( x=F(h) \) is the water surface profile at time \( t=0 \), then the implicit solution of the equation

\[ \frac{\partial h}{\partial t} + (3\sqrt{gh} - 2\sqrt{gh_0})\frac{\partial h}{\partial x} = 0 \]

will be

\[ x = F(h) + (3\sqrt{gh} - 2\sqrt{gh_0})t \]

where \( F(h) \) is the equation of the water surface at \( t=0 \). In other words, each \( h \) value, initially located at \( x_0 \) move according to

\[ x = x_0 + (3\sqrt{gh} - 2\sqrt{gh_0})t \]
OPEN CHANNEL FLOW: Ritter’s dam break (1892)
Let us try to model the sudden removal of a gate when still water is retained upstream of the dam, with still water with \( U=0 \) and \( h=h_0 \). The bed is rectangular, infinitely wide and dry downstream of the dam.

Let us consider the wave that propagates with negative relative velocity (rarefaction wave or onda di depressione). Provided that the bed downstream is dry, there is not a wave propagating with positive relative velocity. For the rarefaction wave the following relationships are valid:

\[
\begin{align*}
    U(t) &= -2\sqrt{gh} + 2\sqrt{gh_0} \\
    a(t) &= -3\sqrt{gh} + 2\sqrt{gh_0}
\end{align*}
\]

From this equation one sees that if \( h = h_0 \):

- \( a = -\sqrt{gh_0} \) if \( h=0 \)
- \( a = 2\sqrt{gh_0} \)

The continuity equation is

\[
\frac{\partial h}{\partial t} + \left(-3\sqrt{gh} + 2\sqrt{gh_0}\right)\frac{\partial h}{\partial x} = 0
\]

From which one sees that each \( h \) value propagates according to

\[
x = \left(-3\sqrt{gh} + 2\sqrt{gh_0}\right)t \quad \text{if} \quad -\sqrt{gh_0} \leq \frac{x}{t} \leq 2\sqrt{gh_0}
\]
First, let us consider the depth at $x=0$, for $t > 0$

$$x = \left(-3\sqrt{gh} + 2\sqrt{gh_0}\right)t \quad \text{if} \quad -\sqrt{gh_0} \leq \frac{x}{t} \leq 2\sqrt{gh_0}$$

It must be

$$3\sqrt{gh} = 2\sqrt{gh_0} \quad \rightarrow \quad h = \frac{4}{9} h_0$$

But we know that

$$U + 2\sqrt{gh} = 0 + 2\sqrt{gh_0} \quad \rightarrow \quad U = 2\sqrt{gh_0} - 2\sqrt{g\frac{4}{9}h_0} = \frac{2}{3}\sqrt{gh_0}$$

So that the discharge per unit width at the breach is

$$q = Uh = \frac{8}{27} h_0\sqrt{gh_0}$$

For instance, at Cancano dam: $h_0 = 28$ m; $b = 265$ m; $q$ Ritter = 137 m$^3$/s; $Q$ Ritter = 36437 m$^3$/s

In the same way it is possible to obtain the water surface profile $h(x,t)$

$$x = \left(-3\sqrt{gh} + 2\sqrt{gh_0}\right)t \quad \rightarrow \quad h = \frac{1}{g} \left(\frac{2}{3}\sqrt{gh_0} - \frac{x}{3t}\right)^2$$
OPEN CHANNEL FLOW: surge in a channel due to the sudden closure of a gate

Water is moving at steady state with constant velocity and depth in a rectangular, infinitely wide bed. A gate is located at the end of the channel. Let us consider the wave that propagates with negative relative velocity upstream from the gate after its sudden closure. The following relationships are valid

\[
\begin{align*}
U + 2\sqrt{gh} &= \text{Cost} \\
U &= U_0 - 2\sqrt{gh} + 2\sqrt{gh_0} \\
a &= U - \sqrt{gh} \\
a &= U_0 - 3\sqrt{gh} + 2\sqrt{gh_0}
\end{align*}
\]

From a one can get the position of the wave with time. The water height \( h_p \) immediately upstream of the gate (where \( U = 0 \)) can be obtained as

\[
0 = U_0 - 2\sqrt{gh} + 2\sqrt{gh_0} \rightarrow h_p = \frac{1}{g}\left(\frac{U_0}{2} + \sqrt{gh_0}\right)^2 = h_0\left(\frac{Fr_0}{2} + 1\right)^2
\]

Which provides a very good approximation as far as \( Fr < 0.83 \).
Let us consider the Oglio river in the final stretch upstream of Lake Iseo. In the last 10 km its slope (term 3 above) is 0.0007 m/m (and in normal flow one can expect the same slope friction $S_f$). Let us consider a flood during which the average velocity $U$ varies between 1 and 2 m/s in 2 hours. Accordingly the local acceleration (1) is approximately $1/(9.81 \times 7200) = 1.41579E-05$ m/m.

At the same time, one can expect a space variation of 1 m/s in 5 km, so that the convective term (2) can be reckoned as $1/(9.81 \times 5 \times 1000) = 2.03874E-05$ m/m.

Accordingly, in this case the (3) and (4) terms are about 40 times larger than the acceleration terms (1) and (2).

Accordingly, there is space for some simplifications.
OPEN CHANNEL FLOW: Parabolic (or diffusive) wave Model

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{1}{b} \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial}{\partial x}(h + z) &= -S_f
\end{align*}
\]

where the slope friction can be regarded as a function of the cross-section conveyance, \( K \)

\[
S_f = \frac{Q^2}{K^2} = \frac{Q^2}{(k_s R^{2/3} S)^2}
\]

Where \( R \) is the hydraulic radius and \( S \) the cross section bh. If the section is infinitely wide, then:

\[
S_f = \frac{Q^2}{k_s^2 h^{10/3} b^2}
\]

By deriving the first equation with respect to \( x \) and the second with respect to \( t \)

\[
\begin{align*}
\frac{\partial^2 h}{\partial x \partial t} + \frac{1}{b} \frac{\partial^2 Q}{\partial x^2} &= 0 \\
\frac{\partial^2 h}{\partial t \partial x} + \frac{\partial S_f}{\partial t} &= \frac{\partial^2 h}{\partial t \partial x} + \frac{2Q}{K^2} \frac{\partial Q}{\partial t} - \frac{2Q^2}{K^3} \frac{\partial K}{\partial h} \frac{\partial h}{\partial t} = 0
\end{align*}
\]

From which the single equation can be obtained

\[
-\frac{1}{b} \frac{\partial^2 Q}{\partial x^2} + \frac{2Q}{K^2} \frac{\partial Q}{\partial t} - \frac{2Q^2}{K^3} \frac{\partial K}{\partial h} \frac{\partial h}{\partial t} = 0
\]

This equation has yet the derivative of \( h \) with respect to \( t \) that can be eliminated using the continuity equation, in order to obtain a PDE only in \( Q \)

\[
\frac{\partial Q}{\partial t} + \frac{Q}{bK} \frac{\partial Q}{\partial h} - \frac{K^2}{2bQ} \frac{\partial^2 Q}{\partial x^2} = 0
\]

This PDE is a parabolic convection-diffusion equation, like the one that governs heat propagation in a flowing medium.
OPEN CHANNEL FLOW: Parabolic (or diffusive) wave

\[
\frac{\partial Q}{\partial t} + \left( \frac{Q}{bK} \frac{\partial K}{\partial h} \right) \frac{\partial Q}{\partial x} - \left( \frac{K^2}{2bQ} \right) \frac{\partial^2 Q}{\partial x^2} = 0
\]

\[
\frac{\partial Q}{\partial t} + c_p(Q) \frac{\partial Q}{\partial x} - \sigma_p(Q) \frac{\partial^2 Q}{\partial x^2} = 0
\]

\[c_p = \frac{Q}{bK} \frac{\partial K}{\partial h} = \frac{U_h \frac{\partial K}{\partial h}}{K} \quad \text{this is the wave celerity of to this model}\]

\[\sigma_p = \frac{K^2}{2bQ} \quad \text{and this is the wave diffusion coefficient}\]

According to this model, an observer moving with the model celerity will see the wave diminishing its peak according to the diffusion coefficient. The solution of this parabolic model requires due conditions, upstream and downstream. Accordingly, this model can take into account backwater effects from downstream

\[\frac{DQ}{Dt} = \sigma_p(Q) \frac{\partial^2 Q}{\partial x^2}\]

If the cross section is infinitely wide, then

\[c_p = \frac{U_h \frac{\partial K}{\partial h}}{K} = U_h \frac{5}{3h} = \frac{5}{3} U\]
In several cases, the bed slope is much larger than the space variation of $h$. Accordingly, a further simplification can be introduced that, being based only on the continuity equations, is named “kinematic model”:

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{1}{b} \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial z}{\partial x} &= -S_f \\
S_0 &= S_f
\end{align*}
\]

This equation states that during the flood propagation the slope friction is equal to the bed slope, a condition that is typical of normal flow.

As a consequence, the Chezy law must be valid during the flow:

\[Q = K \sqrt{i}\]

By deriving it one obtains:

\[\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial K} \frac{\partial K}{\partial h} \frac{\partial h}{\partial x} = \sqrt{i} \frac{\partial K}{\partial h} \frac{\partial h}{\partial x}\]

And by substituting within the mass conservation equation one gets the kinematic model, whose celerity is:

\[c_k = \frac{\sqrt{i} \frac{\partial K}{\partial h}}{b} = \frac{Q}{bK} \frac{\partial K}{\partial h}\]

When the cross section is infinitely large, then:

\[c_k = \frac{Q}{bK} \frac{\partial K}{\partial h} = \frac{5}{3} U\]

According to this model, the wave does not exhibit any decrease of the peak.
OPEN CHANNEL FLOW: kinematic (or purely convective) wave Model

\[ \frac{\partial h}{\partial t} + \frac{1}{b} \sqrt{i} \frac{\partial K}{\partial h} \frac{\partial h}{\partial x} = 0 \]

There are some peculiarities of this model that are evident from its derivation:

- The x,t space is covered by a single family of characteristic lines, whose local slope is \( c_k \)
- Along each characteristic line \( h \) is constant
- Provided that \( Q=Q(h) \), also \( Q \) is constant along each characteristic line. The higher is \( Q \), the higher is \( c_k \)
- Accordingly, each characteristic line is a straight line
- It requires a single boundary condition, upstream. Accordingly, it can’t take into account backwater effects.
- Due to the dependence of the local slope of the characteristic line on \( Q \), the wave does not attenuate its peak but it gets steeper during propagation.