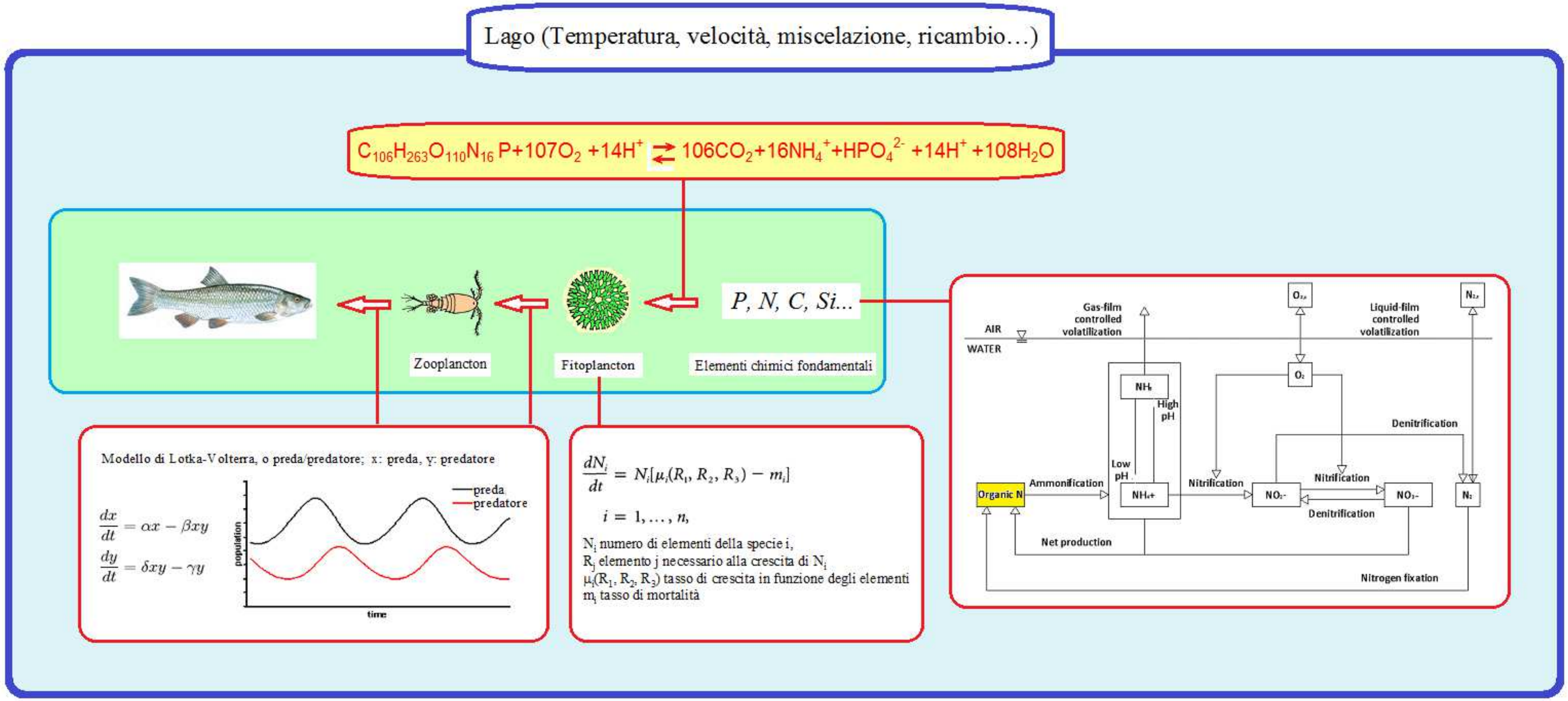
STREETER-PHELPS: Interconnections



Lago (Temperatura, velocità, miscelazione, ricambio…)

$C_{106}H_{263}O_{110}N_{16}P + 107O_2 + 14H^+ \rightleftarrows 106CO_2 + 16NH_4^+ + HPO_4^{2-} + 14H^+ + 108H_2O$

Zooplancton

Fitoplancton

P, N, C, Si…

Elementi chimici fondamentali

Modello di Lotka-Volterra, o preda/predatore; x: preda, y: predatore

$$\frac{dx}{dt} = \alpha x - \beta xy$$

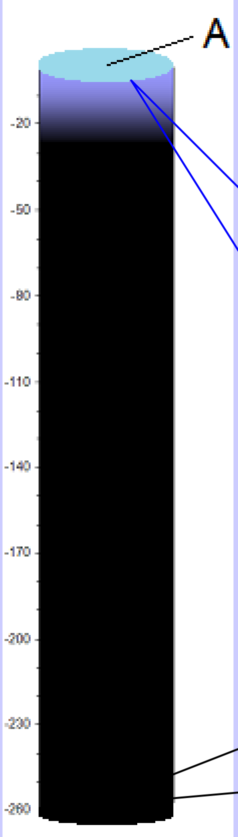
$$\frac{dy}{dt} = \delta xy - \gamma y$$

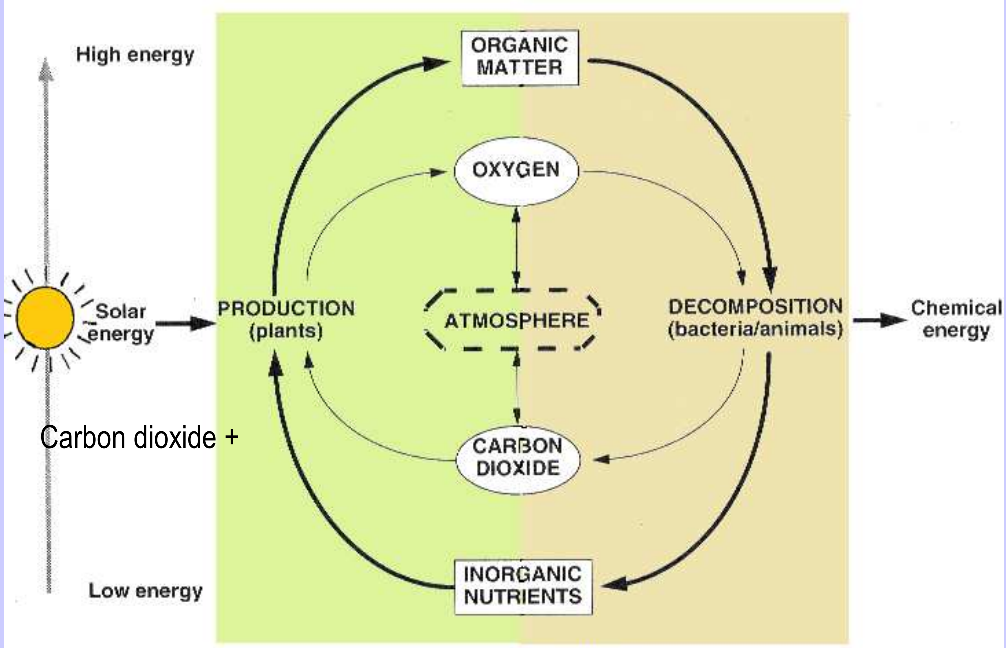
$$\frac{dN_i}{dt} = N_i[\mu_i(R_1, R_2, R_3) - m_i]$$

$$i = 1, \ldots, n,$$

N_i numero di elementi della specie i,
R_j elemento j necessario alla crescita di N_i
$\mu_i(R_1, R_2, R_3)$ tasso di crescita in funzione degli elementi
m_i tasso di mortalità

STREETER-PHELPS: Organic production-decomposition





Adapted from Chapra, S. Surface Water QualityModeling

	Molecular weights
H	1
C	12
N	14
O	16
P	31
S	32
Ca	40

(1)

$$\underset{\text{Carbon dioxide}}{6CO_2}+6H_2O \underset{\text{respiration}}{\overset{\text{Photosyntesis}}{\rightleftarrows}} \underset{\text{sugar}}{C_6H_{12}O_6} + \underset{\text{Oxygen}}{6O_2}$$

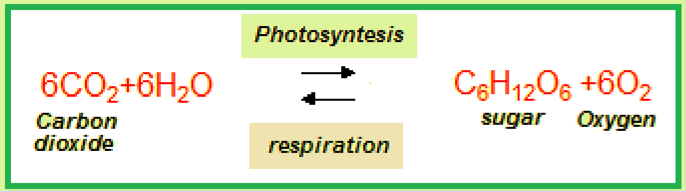
(2)

$$\underset{\text{Carbon dioxide}}{106CO_2}+\overset{\text{Ammonium}}{16NH_4^+}+\underset{\text{Ortophospate}}{HPO_4^{2-}}+14H^+ +108H_2O \underset{\text{respiration}}{\overset{\text{Photosyntesis}}{\rightleftarrows}} \underset{\text{sugar}}{C_{106}H_{263}O_{110}N_{16}P}+\underset{\text{Oxygen}}{107O_2}+14H^+$$

STREETER-PHELPS: Organic production-decomposition

Why Oxygen consumption in water ?

	Molecular weights
H	1
C	12
N	14
O	16



6x32 mg of Oxygen are needed in order to decompose (12x6+1*12+16x6) mg of glucose:

$$r_{OG} = \frac{192}{180} = 1.0667 \frac{mgO}{mgG}$$

Accordingly, if one has 100 g of Glucose, than 106.67 g of Oxygen are needed for its consumption (Note that (6x32)/(106x12+263+110x16+16x14+31)=0.964)

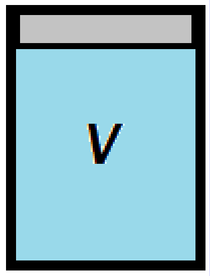
Let us suppose that one can more easily measure C than glucose. One could say that in order to decompose (12x6) mg of C-equivalent sugar, one needs 6x32 mg of Oxygen

$$r_{OC} = \frac{192}{72} = 2.667 \frac{mgO}{mgC}$$

(Note that (6x32)/(106x12)=2.69)

More generally, one does not exactly know the composition of pollutants without a carefull chemical analysis. However one can easily measure experimentally, the overall mg of Oxygen needed to decompose 1 mg of organic pollutant: it’s the **BOD** (Biochemical Oxygen Demand)

STREETER-PHELPS: Organic production-decomposition: Glucose in a CSTR



Example: 4 mg of glucose within a volume V of 1 liter of water in a closed reactor + bacteria+ initial Oxygen concentration O_0 of 8 mg/L. Let us study glucose consumption.

G is the glucose concentration; at t= 0 G=G_0=4 mg/L

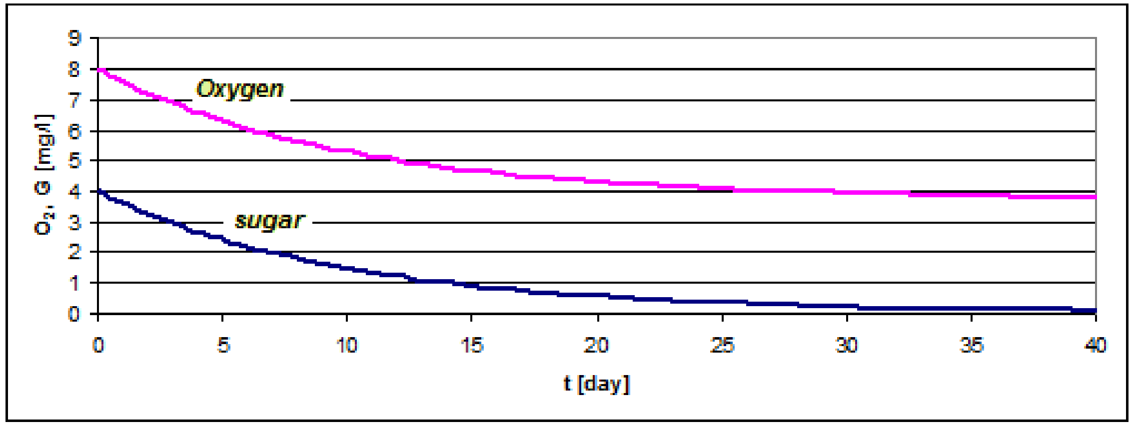
Let us suppose a linear kinetic with constant k = 0.1 [day $^{-1}$]

$$V\frac{dG}{dt} = -kVG; \qquad G = G_0 e^{-kt} \qquad \text{if } \bar{t} = \frac{1}{k} \quad e^{-k\bar{t}} = e^{-1} \cong 0.37$$

Being the bottle closed, Oxygen will decrease according to the kinetic

$$V\frac{dO}{dt} = r_{OG} V \frac{dG}{dt}; \qquad \frac{dO}{dt} = -r_{OG} kG; \qquad \frac{dO}{dt} = -r_{OG} kG_0 e^{-kt};$$

$$O = O_0 - r_{OG} G_0 (1 - e^{-kt})$$



The overall amount of Oxygen needed is r_{OG} G_0= 4.27 mg O^2.

What would happen if we had 10 mg of glucose ?

STREETER-PHELPS: Organic production-decomposition: BOD in a closed CSTR

However, sewage is not made up simply of sugar. It is not practical to think at a detailed analysys of its component and a detailed determination of the decompostion rate. Traditionally, a lumped approach was favoured.

L = amount of Oxidizable matter expressed in terms of Oxygen equivalent [mgO/L], to be determined experimentally. Given that it might be easy to measure $C_{org,}$ the organic carbon concentration, sometimes

$$L = r_{OC} C_{org} = 2.667\, C_{org}$$

BOD(t) = y(t) = L_0 -L(t)   is the Oxygen consumed.

Accordingly, L_0 can be seen either as the initial amount of oxidizable matter or as the final BOD, BOD_∞, when all the organic matter has been consumed.

$$\begin{cases} V\dfrac{dL}{dt} = -k_1 VL \\ \dfrac{dO}{dt} = \dfrac{dL}{dt} \end{cases} \qquad \begin{cases} L = L_0 e^{-k_1 t} \\ \dfrac{dO}{dt} = -k_1 L = -k_1 L_0 e^{-k_1 t} \end{cases} \qquad \boxed{\begin{cases} L = L_0 e^{-k_1 t}; \quad BOD = BOD_\infty \left(1 - e^{-k_1 t}\right) \\ O = O_0 - L_0 (1 - e^{-k_1 t}) \end{cases}}$$

Note that 0.05 <= k_1 <= 0.5 [d^{-1}], but faster for rivers. Accordingly, measuring L_0 or BOD_∞ would take too long, so that usually BOD_5 is measured and BOD_∞ computed as $BOD_\infty = BOD_5 / \left(1 - e^{-5k_1}\right)$

STREETER-PHELPS: BOD and flowing water

Let us consider a flowing body of water studied according to the 1-D approximation that we studied for Open Channel Flow. Let us consider the meaning of the material derivative

$$\frac{DL}{Dt} = \frac{\partial L}{\partial t} + U\frac{\partial L}{\partial x} = 0$$

Water flows without varying the amount of Oxidizable material

$$\frac{DL}{Dt} = -(k_d + k_s)L$$

The amount of Oxidizable material decreases both due to decomposition and due to settling.

The equation above can be rewritten for Steady State as

$$U\frac{\partial L}{\partial x} = -(k_d + k_s)L$$

Accordingly, if one knows the initial concentration at x =0, that is the junction point between the water treatment plant and the river coming from upstream

$$L_0 = \frac{L_{WTP}Q_{WTP} + L_R Q_R}{Q_{WTP} + Q_R}$$

One can easily obtain the L(x) profile along the river $\boxed{L(x) = L_0 e^{-(k_d + k_s)x}}$

This equation can be used also for evaluating the removal rate, because one has simply to measure L(x) as a function of distance

$$\boxed{(k_d + k_s) = \frac{1}{x}\ln\frac{L_0}{L(x)}}$$

STREETER-PHELPS: BOD and flowing water with reareation

Let us consider a flowing body of water studied according to the 1-D approximation that we studied for Open Channel Flow. Let us now consider Oxygen reareation. Here we shall make use of the symbol d/dt but we make reference to the lagrangian meaning D/Dt

$$\frac{dL}{dt} = -(k_d + k_s)L = -k_r L$$

The amount of Oxidizable material decreases due to decompositiona and settling

$$\frac{dO}{dt} = -k_d L \boxed{+k_a}(O_S - O)$$

$$k_a = 3.93\frac{U^{0.5}}{Y^{1.5}}\ [d^{-1}]$$

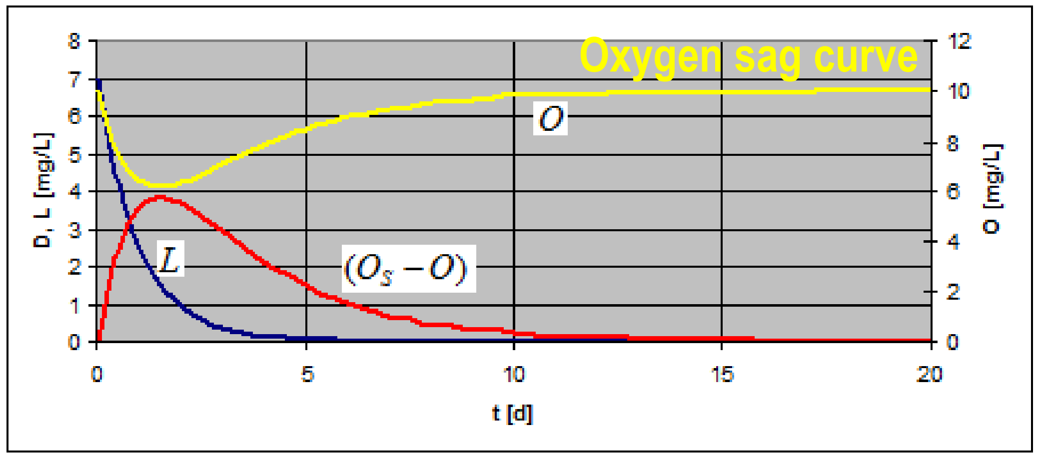
Let us consider Oxygen dynamics: note that O decreases only due to decomposition (k_d in place of k_r !); On the other hand we have a reareation as (e.g. O'Connor-Dobbins formula, with U [m/s], Y [m])

$$D = (O_S - O); \quad dD = -dO$$

the problem analitically simplifies if we introduce the Oxygen deficit D; let us suppose that at station 0, $L=L_0$ and $D = D_0$

$$\begin{cases} \dfrac{dL}{dt} = -k_r L \\ \dfrac{dD}{dt} = +k_d L - k_a D \end{cases} \qquad \begin{cases} L = L_0 e^{-k_r t} \\ \dfrac{dD}{dt} + k_a D = +k_d L_0 e^{-k_r t} \end{cases}$$

$$\boxed{\begin{cases} L(t) = L_0 e^{-k_r t} \\ D(t) = \dfrac{k_d L_0}{k_a - k_r}(e^{-k_r t} - e^{-k_a t}) + D_0 e^{-k_a t} \end{cases}}$$



STREETER-PHELPS: BOD and flowing water with reareation

If we keep in mind that the time derivative from which we started are substantial derivative, we can immediately pass from time to space

$$\begin{cases} L(t) = L_0 e^{-k_r t} \\ D(t) = \dfrac{k_d L_0}{k_a - k_r}(e^{-k_r t} - e^{-k_a t}) + D_0 e^{-k_a t} \end{cases}$$

If U, the water velocity, is constant, then t=x/U, so that the space distribution of L and D is

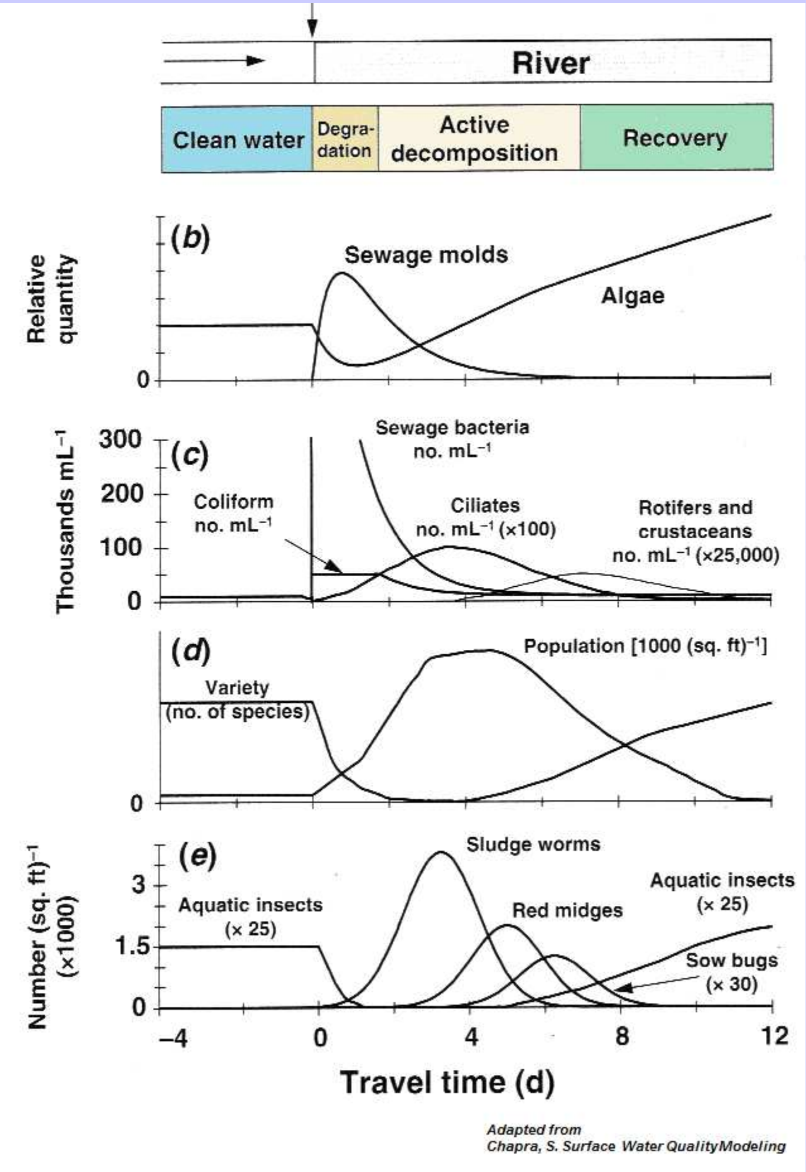
$$\begin{cases} L(x) = L_0 e^{-k_r \frac{x}{U}} \\ D(x) = \dfrac{k_d L_0}{k_a - k_r}(e^{-k_r \frac{x}{U}} - e^{-k_a \frac{x}{U}}) + D_0 e^{-k_a \frac{x}{U}} \end{cases}$$

Finally, if u=u(x) and U is the space average velocity $U = \frac{1}{L}\int_0^L u dx$ then x =tU and $t = \dfrac{xL}{\int_0^L u dx}$

$$\begin{cases} L(t) = L_0 e^{-k_r \frac{xL}{\int_0^L u dx}} \\ D(t) = \dfrac{k_d L_0}{k_a - k_r}(e^{-k_r \frac{xL}{\int_0^L u dx}} - e^{-k_a \frac{xL}{\int_0^L u dx}}) + D_0 e^{-k_a \frac{xL}{\int_0^L u dx}} \end{cases}$$

But if u=u(x) possibly also the coefficients vary in space so that a numerical solution is advisable

STREETER-PHELPS: effects on the environment of sewage treatment plant effluent



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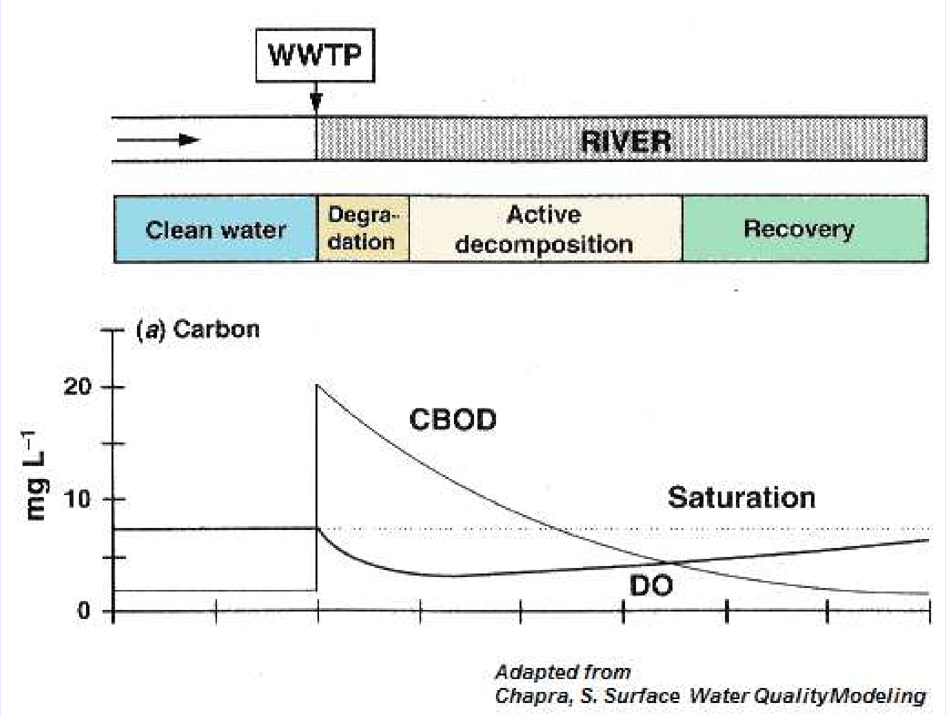
Chapra, S. Surface Water QualityModeling



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Chapra, S. Surface Water QualityModeling

STREETER-PHELPS: BOD and flowing water with reareation

Accordingly, there is a critical station where D is maximum and Oxigen content minimum. This happens in correspondence of a critical travel time, t_c

$$D(t) = \frac{k_d L_0}{k_a - k_r}(e^{-k_r t} - e^{-k_a t}) + D_0 e^{-k_a t}$$

that can be easily obtained by setting $\frac{dD(t)}{dt} = 0$ so obtaining

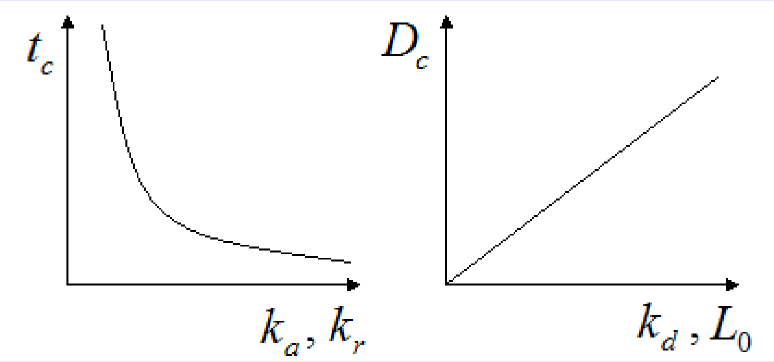
$$\boxed{t_c = \frac{1}{k_a - k_r} \ln\left\{ \frac{k_a}{k_r} \left[1 - \frac{D_0 (k_a - k_r)}{k_d L_0} \right] \right\}}$$

By substituting within D(t) one gets the critical deficit as

$$\boxed{D_c = \frac{k_d L_0}{k_a} \left\{ \frac{k_a}{k_r} \left[1 - \frac{D_0 (k_a - k_r)}{k_d L_0} \right] \right\}^{-\frac{k_r}{k_a - k_r}}}$$

The analysis is simplified if D_0=0

$$\boxed{t_c = \frac{1}{k_a - k_r} \ln\left(\frac{k_a}{k_r} \right); \qquad D_c = \frac{k_d L_0}{k_a} \left\{ \frac{k_a}{k_r} \right\}^{-\frac{k_r}{k_a - k_r}}}$$

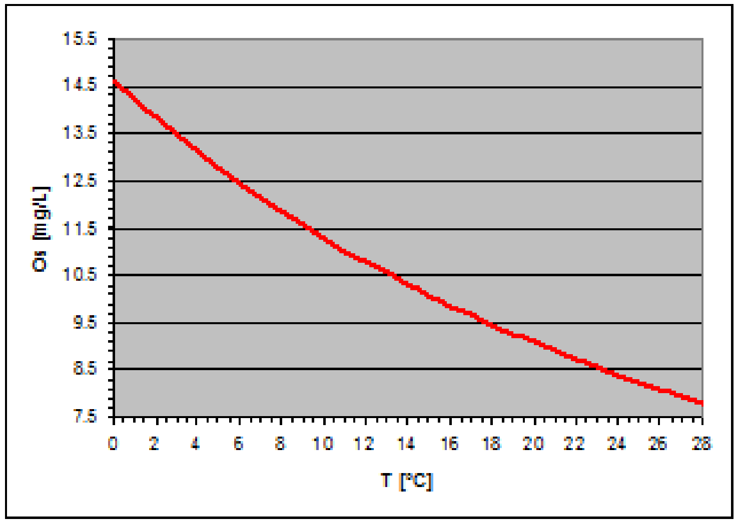


STREETER-PHELPS: BOD and flowing water

D has been introduced to simplify the equation: D= O_s-O. However, O_s is a function of several variables, most noticeably of T. In fresh water at 1 ATM, the relationship holds (APHA, 1992)

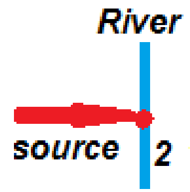
$$\ln(O_s) = -139.34411 + \frac{1.575701*10^5}{T_a} - \frac{6.642308*10^7}{T_a^2} + \frac{1.2438*10^{10}}{T_a^3} - \frac{8.621949*10^{11}}{T_a^4}$$

Where T is in Kelvin.



Accordingly, when using Streeter Phelps model, one has to pay attention at junctions, where an Oxygen balance must be accomplished

	Source	river
Q (m3/s)	0.463	5.787
T [°C]	28	20
DO [mg/L]	2	7.5
DO sat. [mg/L]	7.827	9.092
DO deficit [mg/L]	5.827	1.592



$$T_2 = \frac{0.463 \cdot 28 + 5.787 \cdot 20}{0.463 + 5.787} = 20.59\ °C$$

$$O_2 = \frac{0.463 \cdot 2 + 5.787 \cdot 7.5}{0.463 + 5.787} = 7.093\ mg/L$$

$$O_{S2}(20.59) = 8.987\ mg/L; \qquad DO_{deficit\ 2} = 8.987 - 7.093\ mg/L$$